principal axis for the body. It should be mentioned that since we are neither assembling nor solving a system of equations, the large run times and massive storage requiremens normally associated with finite element analysis are not present in this case. Hence, a mass properties code of the form outlined above can easily be implemented on a minicomputer.

To illustrate the accuracy of the procedure, consider the simple problem of employing plate elements to determine the area properties for a circular section of unit radius. Two such meshes are shown in Figs. 2a and 2b. For the coarser five-element mesh (Fig. 2a) the error in area and moment of inertia are, respectively, 1.178% and 2.338%. While for the finer nine-element mesh (Fig. 2b), the corresponding errors are 0.078% for the area and 0.156% for the moment of inertia. For either solution less than 1 s of CPU time was required on a UNIVAC 1108.

Conclusion

Presented in this Note is a procedure for the accurate evaluation of mass properties for a space body. The procedure is particularly effective when the geometry of the body is irregular. The methodology employed is a perturbation to the standard implementation of iso-parametric finite elements.

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Improved Method for Solving the Algebraic Riccati Equation

W.E. Holley* and S.Y. Wei†
Oregon State University, Corvallis, Ore.

Introduction

SINCE linear quadratic regulator theory came into vogue in the early 1960's, many authors have suggested methods to solve the resulting algebraic Riccati equation. One of the methods that has been most successful is the eigenvector decomposition method first proposed by MacFarlane[1] and by Potter.[2] In this method, the eigenvalues and eigenvectors of the Euler-Lagrange system are determined. The eigenvectors associated with eigenvalues whose real parts are all of the same sign are partitioned into two matrices. These matrices form a set of linear equations which yield the Riccati equation solution.

The success of the method hinges on the requirement that the partitioned eigenvector matrices be nonsingular. In the case when one or more of the eigenvalues is repeated, the resulting matrices may be singular. The singularity can be removed by using the generalized eigenvectors. However, this method is not entirely satisfactory. In the case when the eigenvalues are nearly equal, the partitioned eigenvector matrices are not singular, but they remain ill conditioned. This ill conditioning can lead to errors in the computed solution. Also, small perturbations in the system matrix elements can lead to drastic changes in the partitioned eigenvector matrices, which in turn causes poor numerical stability.

In order to alleviate these difficulties, the following method is proposed:

1) Determine the coefficient matrix H of the Euler-Lagrange system. For the algebraic Riccati equation,

$$SA+A^TS+C^TC-SBB^TS=0$$

the matrix H is given by

$$H=\begin{bmatrix} A, & -BB^T \\ -C^TC, & -A^T \end{bmatrix}$$

2) Calculate the eigenvalues of the coefficient matrix using the highly stable QR algorithm.[3,4]

3) Again use the QR algorithm with shifts of origin beginning at the previously computed eigenvalues with positive real parts. This procedure will compose the system into the form

$$H=P^T\begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}P \tag{1}$$

where

$$P=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is orthogonal and U_{22} has eigenvalues with positive real parts.

4) The symmetric non-negative solution, S, of the Riccati equation is found by solving the linear equation (see Theorem 1 in Appendix)

$$P_{11}S=P_{12} \tag{2}$$

The matrices P_{11} and P_{12} are nonsingular when the system is controllable and observable (see Theorem 2 in Appendix).

If the system has unobservable or uncontrollable modes, then varying the control will not affect the performance index through this part of the system. Thus, these modes can be ignored and the design can proceed using only the completely observable and controllable part of the system.

The major advantage of the method lies in the inherent orthogonality of the P matrix. The QR algorithm produces an extremely stable decomposition into the form of Eq. (1). The P matrix is also not overly sensitive to small perturbations in the system matrix elements or to small perturbations in the computed eigenvalues.

A second advantage of the method is that less computation is required to determine the P matrix than is required to determine the matrix of eigenvectors.

Results

To exemplify the method, consider the following simple optimal control problem:

$$\min_u J=\int_0^\infty\left(x^T\begin{bmatrix} 0 & 0 \\ 0 & q^2 \end{bmatrix}x+u^2\right)dt$$

subject to

$$\dot{x}=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}x+\begin{bmatrix} 0 \\ 1 \end{bmatrix}u \tag{4}$$

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*Assistant Professor. Member AIAA.
†Research Assistant. Member AIAA.

The optimal control law is given by

$$u = -[0 \quad 1]S \tag{5}$$

where S is the symmetric, positive definite solution to the algebraic Riccati equation

$$0 = S\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} S + \begin{bmatrix} 0 & 0 \\ 0 & q^2 \end{bmatrix} - S\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} S \tag{6}$$

The solution is found to be

$$S = \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} \tag{7}$$

The closed-loop characteristic equation is given by

$$s^2 + qs + 1 = 0 \tag{8}$$

Obviously, when $q=2$ the roots of the characteristic equation are equal.

The solution to this problem was attempted using the OPTSYS program described in Bryson and Hall.[5] This program is an implementation of the MacFarlane-Potter approach and usually gives accurate results for fairly high-order systems. However, when a value of $q^2 = 4$ was used, the progam failed completely. Also, when $q^2 = 3.99999$, failure occurred. However, when a value of $q^2 = 4.000001$ was used, the following solution was obtained:

$$S = \begin{bmatrix} 2.0000\text{E}+00 & -1.4552\text{E}-11 \\ 2.9104\text{E}-11 & 2.0000\text{E}+00 \end{bmatrix}$$

When the computed solution was used in the algebraic Riccati Eq. (6), the following residual was obtained:

$$\begin{bmatrix} 2.9104\text{E}-10 & -1.0188\text{E}-09 \\ -1.0186\text{E}-09 & 2.1430\text{E}-10 \end{bmatrix}$$

The method proposed in this paper was implemented on the CDC CYBER 70/73 system, and the following results were obtained with $q^2 = 4$:

$$S = \begin{bmatrix} 2.000\text{E}+00 & 4.6870\text{E}-13 \\ 2.1449\text{E}-13 & 2.0000\text{E}+00 \end{bmatrix}$$

A residual of

$$\begin{bmatrix} -6.8319\text{E}-13 & -7.6687\text{E}-13 \\ -2.5845\text{E}-13 & -5.4001\text{E}-13 \end{bmatrix}$$

resulted.

Conclusion

As can be seen from the results, the proposed method appears to be more accurate and certainly is more stable than the usual MacFarlane-Potter algorithm. It is hoped that the accuracy improvements obtained using the new method will facilitate solutions for systems of higher order than currently available methods can handle.

Appendix—Theoretical Basis of the Algorithm

Lemma 1:[6] The algebraic Riccati equation

$$SA + A^TS + C^TC - SBB^TS = 0 \tag{A1}$$

has a unique positive definite solution S if the matrices (A,B) are controllable and (A,C) are observable. Also the matrix $A\text{-}BB^TS$ has eigenvalues in the open left half-plane.

Lemma 2:[1] Under the conditions of Lemma 1 the eigenvalues of the matrix $A\text{-}BB^TS$ are the same as the left half-plane eigenvalues of the Euler-Langrange system matrix

$$H = \begin{bmatrix} A & -BB^T \\ -C^TC & -A^T \end{bmatrix}$$

Lemma 3:[7] There exists an orthogonal transformation

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

which transforms the matrix H into quasi-upper triangular form

$$\begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} A & -BB^T \\ -C^TC & -A^T \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}^T \tag{A2}$$

From Lemma 1 and 2 the P matrix can be chosen so that U_{11} has all eigenvalues in the left half-plane and U_{22} has all eigenvalues in the right half-plane.

Lemma 4:[8] The solution X to the linear matrix equation

$$AX - XB = 0$$

is unique and equal to zero if A and B have no common eigenvalues.

Theorem 1: Under the conditions of Lemma 1, namely that (A,B) is controllable and (A,C) is observable, the positive definite solution S to the algebraic Riccati Eq. (A1) satisfies the relation

$$P_{11}S = P_{12} \tag{A3}$$

where P_{11} and P_{12} satisfy Eq. (A2).

Proof: Premultiplying Eq. (A2) by $P^T = P^{-1}$ yields

$$\begin{bmatrix} P_{11}^T & P_{21}^T \\ P_{12}^T & P_{22}^T \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} = \begin{bmatrix} A & -BB^T \\ -C^TC & -A^T \end{bmatrix} \begin{bmatrix} P_{11}^T & P_{21}^T \\ P_{12}^T & P_{22}^T \end{bmatrix} \tag{A4}$$

Expanding yields four equations, two of which are

$$AP_{11}^T - BB^TP_{12}^T = P_{11}^TU_{11} \tag{A5}$$

$$-C^TCP_{11}^T - A^TP_{12}^T = P_{12}^TU_{11} \tag{A6}$$

Solving the Riccati Eq. (A1) for C^TC yields

$$C^TC = SBB^TS - SA - A^TS \tag{A7}$$

Using Eq. (A7) in (A6) gives

$$-(SBB^TS - SA - A^TS)P_{11}^T - A^TP_{12}^T = P_{12}^TU_{11} \tag{A8}$$

Using Eq. (A5) in (A8) yields

$$-SBB^TSP_{11}^T + S(P_{11}^T U_{11} + BB^T P_{12}^T) + A^T SP_{11}^T - A^T P_{12}^T = P_{12}^T U_{11}$$

which can be rearranged to give

$$(A - BB^T S)^T (SP_{11}^T - P_{12}^T) + (SP_{11}^T - P_{12}^T) U_{11} = 0 \quad \text{(A9)}$$

From Lemmas 1, 3, and 4,

$$SP_{11}^T - P_{12}^T = 0 \quad \text{or} \quad P_{11}S = P_{12} \quad \text{Q.E.D.}$$

Theorem 2: Under the same conditions as Theorem 1, the matrices P_{11} and P_{12} in Eq. (A3) are nonsingular.

Proof: Expanding Eq. (A2) yields four equations, one of which is

$$U_{11} = P_{11}AP_{11}^T - P_{11}BB^T P_{12}^T - P_{12}C^T CP_{11}^T - P_{12}A^T P_{12}^T \quad \text{(A10)}$$

Using Eqs. (A7) and (A3) in (A10) gives

$$U_{11} = P_{11}(I + S^2)(A - BB^T S) P_{11}^T \quad \text{(A11)}$$

Since U_{11}, S, and A-BB^TS are nonsingular, P_{11} must also be nonsingular. In addition, $P_{12} = P_{11}S$ is nonsingular.

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